L-functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ in the case of high GL_2 conductor

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Abstract. Furusawa [5] has given an integral representation for the degree 8 L-function of $\mathrm{GSp}_4 \times \mathrm{GL}_2$ and has carried out the unramified calculation. The local p-adic zeta integrals were calculated in the work [6] under the assumption that the GSp_4 representation π is unramified and the GL_2 representation τ has conductor $\mathfrak{p}$. In the present work we generalize to the case where the GL_2 representation has arbitrarily high conductor. The result is that the zeta integral represents the local Euler factor $L(s, \pi \times \tau)$ in all cases. As a global application we obtain a special value result for a $\mathrm{GSp}_4 \times \mathrm{GL}_2$ global L-function coming from classical holomorphic cusp forms with arbitrarily high level for the elliptic modular form.

1 Introduction

Let $\pi = \otimes \pi_\nu$ and $\tau = \otimes \tau_\nu$ be irreducible, cuspidal, automorphic representations of $\mathrm{GSp}_4(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{A})$, respectively. Here, $\mathbb{A}$ is the ring of adeles of a number field F. We investigate the degree eight twisted L-functions $L(s, \pi \times \tau)$ of π and τ, which are important for a number of reasons. For example, when π and τ are obtained from holomorphic modular forms, then Deligne [3] has conjectured that a finite set of special values of $L(s, \pi \times \tau)$ are algebraic up to certain period integrals. Another very important application is the conjectured Langlands functorial transfer of π to an automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. One approach to obtain the transfer to $\mathrm{GL}_4(\mathbb{A})$ is to use the converse theorem due to Cogdell and Piatetski-Shapiro [2], which requires precise information about the L-functions $L(s, \pi \times \tau)$.

In [5], Furusawa has obtained an integral representation for $L(s, \pi \times \tau)$ in the special case where π and τ correspond to holomorphic cusp forms of full level and same weight. In [6], we extended Furusawa's integral representation to the case where τ corresponds to a cusp form (holomorphic or non-holomorphic) of square-free level. At the non-archimedean place ν dividing the level, this means that τ_ν is mildly ramified, namely, it is an unramified twist of the Steinberg representation of GL_2.

In this paper, we will compute the local non-archimedean integral for any irreducible, admissible, generic representation τ_ν with unramified central character; the conductor of τ_ν can be arbitrarily high. In the local case, we obtain the following result.

Theorem 1. *Let F_ν be a non-archimedean local field with characteristic zero. Let π_ν be an unramified, irreducible, admissible representation of $\mathrm{GSp}_4(F_\nu)$. Let τ_ν be an irreducible, admissible, generic representation of $\mathrm{GL}_2(F_\nu)$ with unramified central character and conductor $\mathfrak{p}^n, n \geq 2$. Then we can make a choice of vectors such that the local integral (defined in (3)) is given by*

$$Z_\nu(s) = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}(1 - \Big(\frac{L}{\mathfrak{p}}\Big)q^{-1})q^n.$$

Here, q is the cardinality of the residue class field of F_ν and L_ν is the degree 2 extension of F_ν defined in the next section.

The $n = 0$ case was done in [5] and the $n = 1$ case was done in [6]. Note that, for representations π_ν and τ_ν as described in the above theorem, we have $L(s, \pi_\nu \times \tau_\nu) = 1$, and hence the integral $Z_\nu(s)$ indeed computes the L-function up to a constant. Not surprisingly, the case $n > 1$ is not a straightforward generalization of the case $n = 1$, but requires different arguments. Making the "correct" choice of local vectors to be used to compute the local integral is delicate and, probably, is the main contribution of this paper. For example, we will have to make a choice of local compact subgroup $K^\#(\mathfrak{P}^n)$, for which the Borel congruence subgroup

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turns out to be too small, while the Klingen congruence subgroup is too large; the group we will work with lies in between these two natural congruence subgroups. We would like to point out that, in general, the ramified calculation is very complicated in most situations and is rarely carried out in much of the available work on integral representations of L-functions.

We will now describe the global case.

Theorem 2. *Let Φ be a Siegel cuspidal eigenform of weight l with respect to $\mathrm{Sp}_4(\mathbb{Z})$ satisfying the two assumptions described in Section 7. Let N be any positive integer. Let f be a Maaß Hecke eigenform of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$. If f lies in a holomorphic discrete series with lowest weight l_2, then assume that $l_2 \leq l$. Let π_Φ and τ_f be the corresponding cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, respectively. Then a choice of local vectors can be made such that the global integral $Z(s)$ defined in (76) is given by*

$$Z(s) = \kappa_\infty \kappa_N \frac{L(3s+\frac{1}{2}, \pi_\Phi \times \tau_f)}{\zeta(6s+1)L(3s+1, \tau_f \times \mathcal{AI}(\Lambda))},$$

where κ_∞, κ_N are obtained from the local computations, Λ is the Bessel character defined in the next section and $\mathcal{AI}(\Lambda)$ is the representation of $\mathrm{GL}_2(\mathbb{A})$ obtained by automorphic induction from the character Λ.

We can obtain a special value result in the spirit of Deligne's conjectures from the above theorem. For this we need to make an additional assumption on N. Suppose D is the positive integer as in the Assumption 1 on Φ and p is a prime that divides D, then we assume that the highest power of p dividing N is not equal to 2.

Theorem 3. *Let Φ be a Siegel cuspidal eigenform of weight l with respect to $\mathrm{Sp}_4(\mathbb{Z})$. Let N be any positive integer satisfying the condition described above. Let Ψ be a holomorphic cuspidal eigenform of weight l with respect to $\Gamma_0(N)$. Then*

$$\frac{L(\frac{l}{2}-1, \pi_\Phi \times \tau_\Psi)}{\pi^{5l-8}\langle\Phi,\Phi\rangle\langle\Psi,\Psi\rangle_1} \in \bar{\mathbb{Q}}.$$

We refer the reader to [1], [5], [6] and [7] for related results on special values of the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-functions.

This paper is organized as follows. In Section 2, we describe the notations and general setup for the non-archimedean calculation. This is just a brief outline of a more detailed description given in [6]. In Section 3, we describe the choice of the local vector used for the integral calculation. The computation of the integral basically involves three main things – a suitable double coset decomposition, computation of the volumes of the double cosets and computing the expression obtained by substituting the local vectors in the integral. This is carried out in Sections 4, 5 and 6. In Section 7, we put together the local non-archimedean result with the computations from [5] and [6] to obtain the global theorem and the special value result.

2 Preliminaries

In this section, we will briefly recall the setup and terminology in the non-archimedean local case from [6].

i) Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$, $\mathfrak{p}$, ϖ, q be the ring of integers, prime ideal, uniformizer and cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$, respectively.

ii) Fix three elements $a, b, c \in F$ such that $d := b^2 - 4ac \neq 0$. Let $L = F(\sqrt{d})$ if $d \notin F^{\times 2}$ and $L = F \oplus F$ if $d \in F^{\times 2}$. If L is a field, then let $\mathfrak{o}_L$ be its ring of integers. If $L = F \oplus F$, then let $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$. If L is a field, we denote by $\bar{x}$ the Galois conjugate of $x \in L$ over F. If $L = F \oplus F$, let $\overline{(x,y)} = (y,x)$.

iii) We shall assume that $a, b \in \mathfrak{o}$ and $c \in \mathfrak{o}^\times$. In addition, we will assume that if $d \notin F^{\times 2}$, then d is the generator of the discriminant of L/F and if $d \in F^{\times 2}$, then $d \in \mathfrak{o}^\times$.

iv) Let $\alpha := \frac{b+\sqrt{d}}{2c}$ if L is a field and $\alpha := \left(\frac{b+\sqrt{d}}{2c}, \frac{b-\sqrt{d}}{2c}\right)$ if $L = F \oplus F$. Let $\eta = \begin{bmatrix} 1 & 0 & & \\ \alpha & 1 & & \\ & & 1 & -\bar{\alpha} \\ & & 0 & 1 \end{bmatrix}$.

v) We fix the following ideal in $\mathfrak{o}_L$,

$$\mathfrak{P} := \mathfrak{p}\mathfrak{o}_L = \begin{cases} \mathfrak{p}_L & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \mathfrak{p}_L^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \mathfrak{p} \oplus \mathfrak{p} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{1}$$

Here, $\left(\frac{L}{\mathfrak{p}}\right)$ is the Legendre symbol defined in (22) of [6] and $\mathfrak{p}_L$ is the maximal ideal of $\mathfrak{o}_L$ when L is a field extension. Note that $\mathfrak{P}$ is prime only if $\left(\frac{L}{\mathfrak{p}}\right) = -1$. We have $\mathfrak{P}^n \cap \mathfrak{o} = \mathfrak{p}^n$ for all $n \geq 0$.

vi) Let

$$H(F) = \mathrm{GSp}_4(F) := \{g \in \mathrm{GL}_4(F) : {}^t gJg = \mu(g)J,\ \mu(g) \in F^\times\},$$
$$G(F) = \mathrm{GU}(2,2;L) := \{g \in \mathrm{GL}_4(L) : {}^t \bar{g}Jg = \mu(g)J,\ \mu(g) \in F^\times\},$$

where $J = \begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}$. Let the subgroups $P, M^{(1)}, M^{(2)}$ and N of $G(F)$ be as defined in Sect. 2.1 of [6].

vii) For $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ define $T(F) = \{g \in \mathrm{GL}_2(F) : {}^t gSg = \det(g)S\} = \{\begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix} : x, y \in F\} \simeq L^\times$. Let $U(F) = \{\begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix} \in \mathrm{GSp}_4(F) : {}^t X = X\}$. Then the Bessel subgroup $R(F)$ is defined as $R(F) = T(F)U(F)$.

viii) Let $K^H = \mathrm{GSp}_4(\mathfrak{o})$. From (3.4.2) of [5], we have the disjoint double coset decomposition

$$H(F) = \bigsqcup_{l \in \mathbb{Z}} \bigsqcup_{m \geq 0} R(F)h(l,m)K^H, \qquad \text{where } h(l,m) = \begin{bmatrix} \varpi^{2m+l} & & & \\ & \varpi^{m+l} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix}. \tag{2}$$

ix) Fix a nontrivial character ψ of F with conductor $\mathfrak{o}$. Let θ be a character of $U(F)$ obtained from ψ and S as in Sect. 2.2 of [6]. Let Λ be any character of $T(F) \simeq L^\times$. Then we get a character $\Lambda \otimes \theta$ of $R(F)$ as in Sect. 2.2 of [6]. Let (π, V_π) be an unramified, irreducible, admissible representation of $H(F)$ with central character ω_π. Assume that $\Lambda|_{F^\times} = \omega_\pi$. We assume that V_π is a Bessel model for π with respect to the character $\Lambda \otimes \theta$ of $R(F)$. Let B denote the unique spherical vector in V_π satisfying $B(1) = 1$. By Lemma 3.4.4 of [5] we have $B(h(l,m)) = 0$ for $l < 0$.

x) Let (τ, V_τ) be any generic, irreducible, admissible representation of $\mathrm{GL}_2(F)$ with an unramified central character ω_τ. Let χ_0 be a character of $L^\times$ such that $\chi_0|_{F^\times} = \omega_\tau$. Let χ be another character of $L^\times$ such that $\chi(\zeta) = \Lambda(\bar{\zeta})^{-1}\chi_0(\bar{\zeta})^{-1}$. Let $I(s, \chi, \chi_0, \tau)$ be the induced representation of $G(F)$ constructed in Sect. 2.3 of [6]. Further below we will construct a function $W^\# \in I(s, \chi, \chi_0, \tau)$. Our main local result will be the evaluation of the integral

$$Z(s) = \int\limits_{R(F)\backslash H(F)} W^\#(\eta h, s)B(h)dh. \tag{3}$$

3 The local compact group and the function $W^\#$

We define congruence subgroups of $\mathrm{GL}_2(F)$, as follows. For $n=0$ let $K^{(1)}(\mathfrak{p}^0)=\mathrm{GL}_2(\mathfrak{o})$. For $n>0$ let

$$K^{(1)}(\mathfrak{p}^n)=\mathrm{GL}_2(F)\cap\begin{bmatrix}\mathfrak{o}^\times & \mathfrak{o}\\ \mathfrak{p}^n & \mathfrak{o}^\times\end{bmatrix}. \tag{4}$$

Note that we are only considering representations τ with unramified central character. For such representations, the conductor (or level) of τ can be defined in terms of the congruence subgroups (4). More precisely, if n is the smallest integer for which V_τ has a vector that is invariant under $K^{(1)}(\mathfrak{p}^n)$, then we say that $\mathfrak{p}^n$ is the conductor of τ. The space of such invariant vectors is one dimensional. We assume that V_τ is the Whittaker model of τ with respect to the character of F given by $\psi^{-c}(x)=\psi(-cx)$. Note that this character has conductor $\mathfrak{o}$ by our assumptions on ψ and c. Let $W^{(0)}\in V_\tau$ be the local newform, i.e., the essentially unique non-zero $K^{(1)}(\mathfrak{p}^n)$ invariant vector in V_τ. We can make it unique by requiring that $W^{(0)}(1)=1$.

Let $\mathfrak{P}=\mathfrak{p}\mathfrak{o}_L$, as above. Let

$$I:=\{g\in \mathrm{GU}(2,2;\mathfrak{o}_L):\ g\equiv\begin{bmatrix}* & 0 & * & *\\ * & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & 0 & *\end{bmatrix}\pmod{\mathfrak{P}}\} \tag{5}$$

be the Iwahori subgroup and

$$\mathrm{Kl}(\mathfrak{P}^n):=\{g\in \mathrm{GU}(2,2;\mathfrak{o}_L):\ g\equiv\begin{bmatrix}* & 0 & * & *\\ * & * & * & *\\ * & 0 & * & *\\ 0 & 0 & 0 & *\end{bmatrix}\pmod{\mathfrak{P}^n}\} \tag{6}$$

be the Klingen congruence subgroup. We define $K^\#(\mathfrak{P}^0):=\mathrm{GU}(2,2;\mathfrak{o}_L)$ and for $n\geq 1$

$$K^\#(\mathfrak{P}^n):=I\cap \mathrm{Kl}(\mathfrak{P}^n)=\mathrm{GU}(2,2;\mathfrak{o}_L)\cap\begin{bmatrix}\mathfrak{o}_L^\times & \mathfrak{P}^n & \mathfrak{o}_L & \mathfrak{o}_L\\ \mathfrak{o}_L & \mathfrak{o}_L^\times & \mathfrak{o}_L & \mathfrak{o}_L\\ \mathfrak{P} & \mathfrak{P}^n & \mathfrak{o}_L^\times & \mathfrak{o}_L\\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{o}_L^\times\end{bmatrix}. \tag{7}$$

Furthermore, let

$$K^\#(\mathfrak{p}^n):=K^\#(\mathfrak{P}^n)\cap \mathrm{GSp}_4(F)=\mathrm{GU}(2,2;\mathfrak{o})\cap\begin{bmatrix}\mathfrak{o}^\times & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o}\\ \mathfrak{o} & \mathfrak{o}^\times & \mathfrak{o} & \mathfrak{o}\\ \mathfrak{p} & \mathfrak{p}^n & \mathfrak{o}^\times & \mathfrak{o}\\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o}^\times\end{bmatrix}. \tag{8}$$

Note that $K^\#(\mathfrak{P})=I$. Also, note that $K^\#(\mathfrak{P}^n)$ is slightly different from the group (with the same name) defined in Sect. 3.3 of [6].

We will now define the specific function $W^\#$ which we will use to evaluate the integral (3). Since a similar definition has been made in Sect. 2.3 of [6], we will omit some details. We first extend $W^{(0)}$ to a function on $M^{(2)}(F)$ via $W^{(0)}(ag)=\chi_0(a)W^{(0)}(g)$ for $a\in L^\times$ and $g\in\mathrm{GL}_2(F)$. Given a complex number s, there exists a unique function $W^\#(\,\cdot\,,s):\ G(F)\to\mathbb{C}$ with the following properties.

i) If $g\notin M(F)N(F)K^\#(\mathfrak{P}^n)$, then $W^\#(g,s)=0$.

ii) If $g=mnk$ with $m\in M(F)$, $n\in N(F)$, $k\in K^\#(\mathfrak{P}^n)$, then $W^\#(g,s)=W^\#(m,s)$.

iii) For $\zeta \in L^\times$ and $\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix} \in M^{(2)}(F)$,

$$W^\#(\begin{bmatrix}\zeta &&& \\ & 1 && \\ && \bar\zeta^{-1} & \\ &&& 1\end{bmatrix}\begin{bmatrix}1 &&& \\ & \alpha && \beta \\ && \mu & \\ & \gamma && \delta\end{bmatrix}, s) = |N(\zeta)\cdot\mu^{-1}|^{3(s+1/2)}\chi(\zeta)\, W^{(0)}(\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}). \tag{9}$$

Here $\mu = \bar\alpha\delta - \beta\bar\gamma$.

As described in Sect. 3.5 of [6], the integral (3) reduces to

$$Z(s) = \sum_{l,m\ge 0}\sum_i B(h(l,m))\, W^\#(\eta h(l,m)s_i, s) \int\limits_{K_{l,m}\backslash K_{l,m}s_iK^\#(\mathfrak{p}^n)} dh, \tag{10}$$

where $K_{l,m} := h(l,m)^{-1}R(F)h(l,m)\cap K^H$ and $\{s_i\}$ is a system of representatives for the double coset space $K_{l,m}\backslash K^H/K^\#(\mathfrak{p}^n)$. We will now follow the three steps outlined in Sect. 3.5 of [6] to obtain a suitable subset $\{s_i'''\}$ of $\{s_i\}$ for which $W^\#(\eta h(l,m)s_i, s) \ne 0$.

4 Double coset decomposition

4.1 The cosets $K^\#(\mathfrak{p}^0)/K^\#(\mathfrak{p}^n)$

We need to determine representatives for the coset space

$$K^\#(\mathfrak{p}^0)/K^\#(\mathfrak{p}^n), \qquad \text{where } K^\#(\mathfrak{p}^0) = K^H = \mathrm{GSp}(4,\mathfrak{o}). \tag{11}$$

Note that this coset space is isomorphic to

$$K_1^\#(\mathfrak{p}^0)/K_1^\#(\mathfrak{p}^n), \qquad \text{where } K_1^\#(\mathfrak{p}^n) = K^\#(\mathfrak{p}^n)\cap\{g\in H(F) : \mu(g)=1\}. \tag{12}$$

Let

$$s_1 = \begin{bmatrix} & 1 && \\ 1 &&& \\ &&& 1 \\ && 1 & \end{bmatrix}, \qquad s_2 = \begin{bmatrix} &&1& \\ &1&& \\ -1&&& \\ &&&1 \end{bmatrix}. \tag{13}$$

It follows from the Bruhat decomposition for $\mathrm{Sp}(4,\mathfrak{o}/\mathfrak{p})$ that

$$K^\#(\mathfrak{p}^0) = K^\#(\mathfrak{p}^1) \sqcup \bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&& \\ x&1&& \\ &&1&-x \\ &&&1\end{bmatrix}s_1K^\#(\mathfrak{p}^1) \sqcup \bigsqcup_{x\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&x& \\ &1&& \\ &&1& \\ &&&1\end{bmatrix}s_2K^\#(\mathfrak{p}^1) \tag{14}$$

$$\sqcup \bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&& \\ x&1&&y \\ &&1&-x \\ &&&1\end{bmatrix}s_1s_2K^\#(\mathfrak{p}^1) \sqcup \bigsqcup_{x,y\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&x&y \\ &1&y& \\ &&1& \\ &&&1\end{bmatrix}s_2s_1K^\#(\mathfrak{p}^1) \tag{15}$$

$$\sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&&y \\ x&1&y&xy+z \\ &&1&-x \\ &&&1\end{bmatrix}s_1s_2s_1K^\#(\mathfrak{p}^1) \sqcup \bigsqcup_{x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&x&y \\ &1&y&z \\ &&1& \\ &&&1\end{bmatrix}s_2s_1s_2K^\#(\mathfrak{p}^1) \tag{16}$$

$$\sqcup \bigsqcup_{w,x,y,z\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1&&x&y \\ w&1&wx+y&wy+z \\ &&1&-w \\ &&&1\end{bmatrix}s_1s_2s_1s_2K^\#(\mathfrak{p}^1). \tag{17}$$

Let $n \geq 1$. It is easy to see that

$$K^{\#}(\mathfrak{p}^1) = \bigsqcup_{w,y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n). \tag{18}$$

Let $\{r_i\}$ be the system of representatives for $K^{\#}(\mathfrak{p}^0)/K^{\#}(\mathfrak{p}^1)$ determined in (14) – (17). Combining these with (18) we get

$$K^H = \bigsqcup_i \bigsqcup_{w,y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} r_i \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n). \tag{19}$$

Recall that we are interested in the double cosets $K_{l,m} \backslash K^H / K^{\#}(\mathfrak{p}^n)$, where $K_{l,m} = h(l,m)^{-1} R(F) h(l,m) \cap K^H$.

4.2 Step 1: Preliminary decomposition

Observe that $K_{l,m}$ contains all elements $\begin{bmatrix} 1 & & \mathfrak{o} & \mathfrak{o} \\ & 1 & \mathfrak{o} & \mathfrak{o} \\ & & 1 & \\ & & & 1 \end{bmatrix}$. From (19) we get the following preliminary decomposition, which is not disjoint.

$$\begin{aligned}
K^H \quad = \quad & \bigcup_{y,z,w \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y\varpi & 1 & \\ y\varpi & z\varpi & & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n) && (20) \\
& \cup \bigcup_{w \in \mathfrak{o}/\mathfrak{p}^n} \bigcup_{y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ z\varpi & y\varpi & 1 & \\ y\varpi & & & 1 \end{bmatrix} s_1 K^{\#}(\mathfrak{p}^n) && (21) \\
& \cup \bigcup_{w,y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & w\varpi & 1 & \\ w\varpi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & y\varpi & & \\ & 1 & & \\ & & 1 & \\ & z\varpi & -y\varpi & 1 \end{bmatrix} s_2 K^{\#}(\mathfrak{p}^n) && (22) \\
& \cup \bigcup_{w \in \mathfrak{o}/\mathfrak{p}^n} \bigcup_{y,z \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ y\varpi & z\varpi & 1 & \\ z\varpi & & & 1 \end{bmatrix} s_1 s_2 K^{\#}(\mathfrak{p}^n) && (23) \\
& \cup \bigcup_{w \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} s_2 s_1 K^{\#}(\mathfrak{p}^n) && (24) \\
& \cup \bigcup_{w \in \mathfrak{o}/\mathfrak{p}^n} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 K^{\#}(\mathfrak{p}^n) && (25) \\
& \cup \bigcup_{w \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} s_2 s_1 s_2 K^{\#}(\mathfrak{p}^n) && (26)
\end{aligned}$$

$$\cup \bigcup_{w\in\mathfrak{o}/\mathfrak{p}^n} K_{l,m}\begin{bmatrix}1&&&\\w&1&&\\&&1&-w\\&&&1\end{bmatrix} s_1s_2s_1s_2K^{\#}(\mathfrak{p}^n). \tag{27}$$

4.3 Step 2: Support of $W^{\#}$

We assumed that $c\in\mathfrak{o}^\times$, so that $\alpha\in\mathfrak{o}_L$. We have $\eta h(l,m)=h(l,m)\eta_m$, where for $m\geq 0$ we let

$$\eta_m=\begin{bmatrix}1&&&\\\alpha\varpi^m&1&&\\&&1&-\bar\alpha\varpi^m\\&&&1\end{bmatrix}. \tag{28}$$

Fix $l,m\geq 0$, and let r run through the representatives for $K_{l,m}\backslash K^H/K^{\#}(\mathfrak{p}^n)$ from (20) – (27). In view of (10) we want to find out for which r is $\eta h(l,m)r\in M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since this set is the support of $W^{\#}$. Since $h(l,m)\in M(F)$, this is equivalent to $\eta_m r\in M(F)N(F)K^{\#}(\mathfrak{P}^n)$. Hence, this condition depends only on $m\geq 0$ and not on the integer l.

i) Let $r=\begin{bmatrix}1&w\varpi&&\\&1&&\\&&1&\\&&-w\varpi&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&y\varpi&1&\\y\varpi&z\varpi&&1\end{bmatrix}$ with $w,y,z\in\mathfrak{o}/\mathfrak{p}^{n-1}$. Suppose $\eta_m r=\tilde m\tilde n k$ with $\tilde m\in M(F)$, $\tilde n\in N(F)$ and $k\in K^{\#}(\mathfrak{P}^n)$. Let $A=(\tilde m\tilde n)^{-1}\eta_m r$. Looking at the $(3,2)$ and $(3,3)$ coefficient of A we get

$$y+\varpi^{m+1}\alpha wy-\varpi^m\alpha z\in\mathfrak{P}^{n-1},\qquad\text{and hence}\qquad \alpha\varpi^m(\varpi wy-z)+y\in\mathfrak{P}^{n-1}.$$

If $\nu(\varpi wy-z)<n-m-1$ then $\alpha+y/(\varpi^m(\varpi wy-z))\in\mathfrak{P}$, which contradicts Lemma 3.1.1 (ii) of [6]. Hence, $\nu(\varpi wy-z)\geq n-m-1$, which implies $\varpi^m(\varpi wy-z)\in\mathfrak{p}^{n-1}$. It follows that $y\in\mathfrak{p}^{n-1}$. To summarize, necessary conditions for $A\in K^{\#}(\mathfrak{P}^n)$ are $y=0$ and $z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}$. The following matrix identity shows that these are also sufficient conditions.

$$\begin{aligned}\eta_m r\;=\;&\begin{bmatrix}a^{-1}&&&\\&a&&\\&&\bar a&\\&\varpi z\bar a^{-1}&&\bar a^{-1}\end{bmatrix}\begin{bmatrix}1&\varpi wa&&\\&1&&\\&&1&\\&&-\varpi w\bar a&1\end{bmatrix}\\&\begin{bmatrix}1&&&\\\varpi^m\alpha a^{-1}&1&&\\&-\varpi^{m+1}\bar\alpha z\bar a^{-1}&1&-\varpi^m\bar\alpha\bar a^{-1}\\-\varpi^{m+1}\alpha z a^{-1}&&&1\end{bmatrix}\in M(F)N(F)K^{\#}(\mathfrak{P}^n),\end{aligned} \tag{29}$$

where $a=1+\varpi^{m+1}\alpha w\in\mathfrak{o}_L^\times$. Hence, the values of w,y,z for which $\eta_m r\in M(F)N(F)K^{\#}(\mathfrak{P}^n)$ are

$$w\in\mathfrak{o}/\mathfrak{p}^{n-1},\ y=0,\ z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}.$$

ii) Let $r=\begin{bmatrix}1&&&\\w&1&&\\&&1&-w\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\z\varpi&y\varpi&1&\\y\varpi&&&1\end{bmatrix}s_1$ with $w\in\mathfrak{o}/\mathfrak{p}^n$ and $y,z\in\mathfrak{o}/\mathfrak{p}^{n-1}$. Suppose $\eta_m r=\tilde m\tilde n k$ with $\tilde m\in M(F),\tilde n\in N(F)$ and $k\in K^{\#}(\mathfrak{P}^n)$. Let $A=(\tilde m\tilde n)^{-1}\eta_m r$. Looking at the $(3,2)$ and $(3,3)$ coefficients of A we get

$$\beta:=\varpi^m\alpha+w\in\mathfrak{o}_L^\times\qquad\text{and}\qquad\varpi^m\alpha y+wy-z\in\mathfrak{P}^{n-1}.$$

If $\nu(y) < n-m-1$, then $\alpha + (wy-z)/(\varpi^m y) \in \mathfrak{P}$, which contradicts Lemma 3.1.1(ii) of [6]. Hence, $\nu(y) \geq n-m-1$, which implies $wy - z \in \mathfrak{P}^{n-1}$. We may therefore assume that $z = wy$. To summarize, necessary conditions for $A \in K^{\#}(\mathfrak{P}^n)$ are $\varpi^m\alpha + w \in \mathfrak{o}_L^\times$, $y \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$ and $z = wy$. The following matrix identity shows that these are also sufficient conditions.

$$\begin{aligned}\eta_m r \quad &= \quad \begin{bmatrix} -\beta^{-1} &&& \\ & \beta && \\ && -\bar\beta & \\ & \varpi wy\bar\beta^{-1} && \bar\beta^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\beta && \\ & 1 && \\ && 1 & \\ && \bar\beta & 1 \end{bmatrix} \\ &\qquad \begin{bmatrix} 1 &&& \\ \beta^{-1} & 1 && \\ -\varpi y\bar\beta^{-1} & \varpi^{m+1}\bar\alpha y\bar\beta^{-1} & 1 & -\bar\beta^{-1} \\ \varpi^{m+1}\alpha y\beta^{-1} &&& 1 \end{bmatrix} \in M(F)N(F)K^{\#}(\mathfrak{P}^n).\end{aligned} \tag{30}$$

Hence, the values of w, y, z for which $\eta_m r \in M(F)N(F)K^{\#}(\mathfrak{P}^n)$ are as follows.

a) If $m = 0$, then all $w \in \mathfrak{o}/\mathfrak{p}^n$ such that $\alpha + w \in \mathfrak{o}_L^\times$ and $y = z = 0$.

b) If $m > 0$, then all $w \in \mathfrak{o}^\times$, $y \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$ and $z = wy$.

iii) Let $r = \begin{bmatrix} 1 &&& \\ & 1 && \\ & w\varpi & 1 & \\ w\varpi &&& 1 \end{bmatrix} \begin{bmatrix} 1 & y\varpi && \\ & 1 && \\ && 1 & \\ & z\varpi & -y\varpi & 1 \end{bmatrix} s_2$ with $w, y, z \in \mathfrak{o}/\mathfrak{p}^{n-1}$.

Then $\eta_m r \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since the $(3,3)$-coefficient divided by the $(3,1)$-coefficient of any matrix product of the form $\tilde n^{-1}\tilde m^{-1}\eta_m r$, $\tilde m \in M(F)$, $\tilde n \in N(F)$, is in $\mathfrak{o}_L$.

iv) Let $r = \begin{bmatrix} 1 &&& \\ w & 1 && \\ && 1 & -w \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ y\varpi & z\varpi & 1 & \\ z\varpi &&& 1 \end{bmatrix} s_1 s_2$ with $w \in \mathfrak{o}/\mathfrak{p}^n$ and $y, z \in \mathfrak{o}/\mathfrak{p}^{n-1}$.

Then $\eta_m r \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since the $(3,3)$-coefficient of any product of the form $\tilde n^{-1}\tilde m^{-1}\eta_m r$, $\tilde m \in M(F)$, $\tilde n \in N(F)$, is in $\mathfrak{P}$.

v) Let $r = \begin{bmatrix} 1 & w\varpi && \\ & 1 && \\ && 1 & \\ && -w\varpi & 1 \end{bmatrix} s_2 s_1$ with $w \in \mathfrak{o}/\mathfrak{p}^{n-1}$.

Then $\eta_m r \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since the $(4,1)$-coefficient of any product of the form $\tilde n^{-1}\tilde m^{-1}\eta_m r$, $\tilde m \in M(F)$, $\tilde n \in N(F)$, is in $\mathfrak{o}_L^\times$.

vi) Let $r = \begin{bmatrix} 1 &&& \\ w & 1 && \\ && 1 & -w \\ &&& 1 \end{bmatrix} s_1 s_2 s_1$ with $w \in \mathfrak{o}/\mathfrak{p}^n$. Suppose $\eta_m r = \tilde m \tilde n k$ with $\tilde m \in M(F), \tilde n \in N(F)$ and $k \in K^{\#}(\mathfrak{P}^n)$. Let $A = (\tilde m \tilde n)^{-1}\eta_m r$. Looking at the $(3,2)$ and $(3,3)$ coefficients of A we get $\varpi^m\alpha + w \in \mathfrak{P}^n$. If $m < n$, then we get $\alpha + w/\varpi^m \in \mathfrak{P}$ which contradicts Lemma 3.1.1 (ii) of [6]. Hence $m \geq n$, which implies that $w \in \mathfrak{P}^n$. We may therefore assume that $w = 0$. To summarize, necessary conditions for $A \in K^{\#}(\mathfrak{P}^n)$ are $m \geq n$ and $w = 0$. The following matrix identity shows that these are also sufficient conditions.

$$\eta_m r = \begin{bmatrix} 1 &&& \\ &&& 1 \\ && 1 & \\ & -1 && \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ & \varpi^m\bar\alpha & 1 & \\ \varpi^m\alpha &&& 1 \end{bmatrix} \in M(F)N(F)K^{\#}(\mathfrak{P}^n). \tag{31}$$

vii) Let $r = \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} s_2 s_1 s_2$ with $w \in \mathfrak{o}/\mathfrak{p}^{n-1}$.

Then $\eta_m r \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since the $(3,3)$-coefficient of any product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is zero.

viii) Let $r = \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} s_1 s_2 s_1 s_2$ with $w \in \mathfrak{o}/\mathfrak{p}^n$.

Then $\eta_m r \notin M(F)N(F)K^{\#}(\mathfrak{P}^n)$, since the $(3,3)$-coefficient of any product of the form $\tilde{n}^{-1}\tilde{m}^{-1}\eta_m r$, $\tilde{m} \in M(F)$, $\tilde{n} \in N(F)$, is zero.

Let us summarize the double cosets that can possibly make a non-trivial contribution to the integral (10).

$$\bigcup_{\substack{w \in \mathfrak{o}/\mathfrak{p}^{n-1} \\ z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}}} K_{l,m} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & z\varpi & & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n) \qquad \text{for } l, m \geq 0, \tag{32}$$

$$\bigcup_{\substack{w \in \mathfrak{o}/\mathfrak{p}^{n} \\ \varpi^m \alpha + w \in \mathfrak{o}_L^\times \\ y \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}}} K_{l,m} \begin{bmatrix} 1 & & & \\ w & 1 & & \\ & & 1 & -w \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ yw\varpi & y\varpi & 1 & \\ y\varpi & & & 1 \end{bmatrix} s_1 K^{\#}(\mathfrak{p}^n) \qquad \text{for } l, m \geq 0, \tag{33}$$

$$K_{l,m} s_1 s_2 s_1 K^{\#}(\mathfrak{p}^n) \qquad \text{for } l \geq 0,\ m \geq n. \tag{34}$$

4.4 Step 3: Disjointness of double cosets

We will now investigate the overlap between double cosets in (32), (33) and (34). First we will consider the case $m = 0$.

Equivalences among double cosets from (32) with $m = 0$

For $w \in \mathfrak{o}/\mathfrak{p}^{n-1}$, set $\beta = c + b(\varpi w) + a(\varpi w)^2 \in \mathfrak{o}^\times$. Let $g = \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix}$ with $y = \varpi w$ and $x = c + yb/2$. Then we have the matrix identity

$$h(l,0)^{-1} \begin{bmatrix} g & \\ & \det(g)\,{}^t g^{-1} \end{bmatrix} h(l,0) = \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} \begin{bmatrix} \beta & & & \\ -a\varpi w & c & & \\ & & c & a\varpi w \\ & & & \beta \end{bmatrix}.$$

The rightmost matrix above is in $K^{\#}(\mathfrak{p}^n)$, so that

$$\bigcup_{w \in \mathfrak{o}/\mathfrak{p}^{n-1}} K_{l,0} \begin{bmatrix} 1 & w\varpi & & \\ & 1 & & \\ & & 1 & \\ & & -w\varpi & 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n) = K_{l,0} K^{\#}(\mathfrak{p}^n) \qquad \text{for all } l \geq 0. \tag{35}$$

Equivalences among double cosets from (35) and (33) with $m = 0$

Let $w \in \mathfrak{o}/\mathfrak{p}^n$ be such that $\alpha + w \in \mathfrak{o}_L^\times$. Set $\beta = a + bw + cw^2$. Let $g = \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix}$ with $y = 1$ and $x = -(cw + b/2)$. Then we have the matrix identity

$$h(l,0)^{-1} \begin{bmatrix} g & \\ & \det(g)\, {}^t g^{-1} \end{bmatrix} h(l,0) \begin{bmatrix} 1 &&& \\ w & 1 && \\ && 1 & -w \\ &&& 1 \end{bmatrix} s_1 = \begin{bmatrix} c &&& \\ -(b+cw) & -\beta && \\ && \beta & -(b+cw) \\ &&& -c \end{bmatrix}$$

The matrix on the right hand side above is in $K^\#(\mathfrak{p}^n)$ if $\beta \in \mathfrak{o}^\times$. We will now show that the condition $\alpha + w \in \mathfrak{o}_L^\times$ forces $\beta \in \mathfrak{o}^\times$. First observe the identity

$$a + bw + cw^2 = -c(\alpha + w)(\alpha - (w + bc^{-1})).$$

If $\beta \in \mathfrak{p}$, then it would follow that $\alpha - (w + bc^{-1}) \in \mathfrak{p}\mathfrak{o}_L = \mathfrak{P}$. By Lemma 3.1.1 (ii) of [6], this is impossible. It follows that indeed $\beta \in \mathfrak{o}^\times$, so that *all* double cosets in (33) with $m = 0$ are equivalent to the double coset in (35).

Equivalence among double cosets from (32) or (33) and (34) with $m > 0$

Let h_1 be a double coset representative obtained in either (32) or (33) and let h_2 be a double coset representative from (34). Then, in either case, the double cosets are not equivalent, since, for any $r \in R(F)$ the $(2,2)$ coordinate of the matrix $h_2^{-1} h(l,m)^{-1} r h(l,m) h_1$ is in $\mathfrak{p}$.

Equivalence among double cosets from (32) and (33) with $m > 0$

For $m > 0$ the condition $\varpi^m \alpha + w \in \mathfrak{o}_L^\times$ in (33) is equivalent to $w \in \mathfrak{o}^\times$. Hence let $w \in \mathfrak{o}^\times$ and $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$. Let $\beta_1 = a\varpi^{2m} + b\varpi^m + c$, $\beta_2 = a\varpi^{2m} + b\varpi^m + cw$ and $\beta_3 = a\varpi^{2m} + bw\varpi^m + cw^2$. We have $\beta_1, \beta_2, \beta_3 \in \mathfrak{o}^\times$. Let $g = \begin{bmatrix} x + yb/2 & yc \\ -ya & x - yb/2 \end{bmatrix}$ with $y = \varpi^m(1-w)/\beta_3$ and $x = \beta_2/\beta_3 - by/2$. Then we have the matrix identity

$$h(l,m)^{-1} \begin{bmatrix} g & \\ & \det(g)\, {}^t g^{-1} \end{bmatrix} h(l,m) \begin{bmatrix} 1 &&& \\ 1 & 1 && \\ && 1 & -1 \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ z\varpi/w & z\varpi/w & 1 & \\ z\varpi/w &&& 1 \end{bmatrix} s_1$$

$$= \begin{bmatrix} 1 &&& \\ w & 1 && \\ && 1 & -w \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ zw\varpi & z\varpi & 1 & \\ z\varpi &&& 1 \end{bmatrix} s_1 \kappa,$$

where

$$\kappa = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{c(1-w)}{\beta_3} & \frac{\beta_1}{\beta_3} & 0 & 0 \\ \frac{cz\varpi(w^2-1)}{w\beta_3} & -\frac{\varpi^{m+1} z(w-1)(b + a\varpi^m)}{w\beta_3} & \frac{\beta_1}{\beta_3} & \frac{c(w-1)}{\beta_3} \\ -\frac{\varpi^{m+1} z(w-1)(bw + a\varpi^m)}{w\beta_3} & -\frac{\varpi^{m+1} z(w-1)(bw + a\varpi^m(1+w))}{w\beta_3} & 0 & 1 \end{bmatrix} \in K^\#(\mathfrak{p}^n).$$

Hence

$$\bigcup_{\substack{w \in \mathfrak{o}/\mathfrak{p}^n \\ w \in \mathfrak{o}^\times \\ z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}}} K_{l,m} \begin{bmatrix} 1 &&& \\ w & 1 && \\ && 1 & -w \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ zw\varpi & z\varpi & 1 & \\ z\varpi &&& 1 \end{bmatrix} s_1 K^\#(\mathfrak{p}^n)$$

$$= \bigcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ 1 & 1 && \\ && 1 & -1 \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ z\varpi & z\varpi & 1 & \\ z\varpi &&& 1 \end{bmatrix} s_1 K^{\#}(\mathfrak{p}^n). \tag{36}$$

Now let $w \in \mathfrak{o}/\mathfrak{p}^{n-1}$ and $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$. Set $\beta = c + (\varpi^{m+1}w)b + (\varpi^{m+1}w)^2 a \in \mathfrak{o}^\times$. Let $g_1 = \begin{bmatrix} x_1 + y_1 b/2 & y_1 c \\ -y_1 a & x_1 - y_1 b/2 \end{bmatrix}$ with $y_1 = \varpi^{m+1}w/\beta$ and $x_1 = 1 - by_1/2 - a\varpi^{m+1}wy_1$. Then we have the matrix identity

$$h(l,m)^{-1} \begin{bmatrix} g_1 & \\ & \det(g_1)\, {}^t(g_1)^{-1} \end{bmatrix} h(l,m) \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} = \begin{bmatrix} 1 & w\varpi && \\ & 1 && \\ && 1 & \\ && -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} \kappa_1,$$

where

$$\kappa_1 = \begin{bmatrix} 1 &&& \\ -a\varpi^{2m+1}w/\beta & c/\beta && \\ & a\varpi^{2+2m}wz/\beta & c/\beta & a\varpi^{2m+1}w/\beta \\ a\varpi^{2+2m}wz/\beta & \varpi^{2+m}w(b + a\varpi^{m+1}w)z/\beta && 1 \end{bmatrix} \in K^{\#}(\mathfrak{p}^n).$$

Hence

$$\bigcup_{\substack{w\in\mathfrak{o}/\mathfrak{p}^{n-1} \\ z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}}} K_{l,m} \begin{bmatrix} 1 & w\varpi && \\ & 1 && \\ && 1 & \\ && -w\varpi & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n)$$
$$= \bigcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n). \tag{37}$$

We will now show that the double cosets in (36) are all equivalent to double cosets in (37). Given $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$, let $g_2 = \begin{bmatrix} x_2 + y_2 b/2 & y_2 c \\ -y_2 a & x_2 - y_2 b/2 \end{bmatrix}$ with $y_2 = \varpi^m$ and $x_2 = -(c + by_2/2)$. Then we have the matrix identity

$$h(l,m)^{-1} \begin{bmatrix} g_1 & \\ & \det(g_1)\, {}^t(g_1)^{-1} \end{bmatrix} h(l,m) \begin{bmatrix} 1 &&& \\ 1 & 1 && \\ && 1 & -1 \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & 1 && \\ z\varpi & z\varpi & 1 & \\ z\varpi &&& 1 \end{bmatrix} s_1 = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} \kappa_2,$$

where

$$\kappa_2 = \begin{bmatrix} c &&& \\ -c - b\varpi^m & -c - b\varpi^m - a\varpi^{2m} && \\ -\varpi(c + b\varpi^m)z & a\varpi^{1+2m}z & c + b\varpi^m + a\varpi^{2m} & -c - b\varpi^m \\ b\varpi^{m+1}z & \varpi^{m+1}(b + a\varpi^m)z & 0 & -c \end{bmatrix} \in K^{\#}(\mathfrak{p}^n).$$

We conclude that, for $m > 0$ and any $l \geq 0$, the double cosets in (32) and (33) are all contained in the union

$$\bigcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n). \tag{38}$$

Equivalence among double cosets from (38) with $m > 0$

Finally, we have to determine any equivalences amongst the double cosets in (38). Fix $l \geq 0$ and $m > 0$, and let

$$h_1 = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z_1\varpi && 1 \end{bmatrix}, \qquad h_2 = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z_2\varpi && 1 \end{bmatrix}$$

with $z_1, z_2 \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$. We want to see if we can find $r = \begin{bmatrix} g & gX \\ & \det(g)\,{}^tg^{-1} \end{bmatrix} \in R(F)$ such that

$$A = h_1^{-1} h(l,m)^{-1} r h(l,m) h_2 \in K^{\#}(\mathfrak{p}^n);$$

here, $g = \begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \in T(F)$ and $X = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$ with $e, f, g \in F$. Suppose such an r exists. Looking at the $(1,3)$, $(1,4)$, $(2,3)$ and $(2,4)$ coefficient of A we get

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{l+2m} & \mathfrak{p}^{l+m} \\ \mathfrak{p}^{l+m} & \mathfrak{p}^{l} \end{bmatrix}.$$

Looking at the $(1,1)$, $(1,2)$, $(1,4)$ and $(3,3)$ coefficient of A, we see that

$$x \pm by/2 \in \mathfrak{o}^\times, y \in \mathfrak{p}^m \qquad \text{and hence} \qquad \begin{bmatrix} e & f \\ f & g \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{l+2m} & \mathfrak{p}^{l+m} \\ \mathfrak{p}^{l+m} & \mathfrak{p}^{l} \end{bmatrix}.$$

Looking at the $(4,2)$ coefficient of A, we get

$$(x-by/2)z_1 + \varpi^{1-l}(g(x-by/2) - afy)z_1z_2 - (x+by/2)z_2 \in \mathfrak{p}^{n-1}. \tag{39}$$

From this it follows that $\nu(z_1) = \nu(z_2)$. Using $y \in \mathfrak{p}^m$, it further follows that

$$(z_1 - z_2) + \varpi^{-l} g(\varpi z_1 z_2) \in \mathfrak{p}^{n-1}. \tag{40}$$

(first add byz_2 to both sides of (39), then divide by the unit $x - by/2$). Let $\nu(z_1) = \nu(z_2) = j$. Write $z_i = \varpi^j u_i$ for $i = 1, 2$, where $u_i \in \mathfrak{o}^\times$. If $2j + 1 \geq n - 1$, then (40) implies that $z_1 = z_2$ which gives us that h_1 and h_2 define disjoint double cosets. If $2j + 1 < n - 1$, then (40) implies that $u_1 - u_2 \in \mathfrak{p}^{j+1}$. This is a necessary condition for the coincidence of double cosets.

We will now show that it is sufficient. So, suppose that $u_1 - u_2 \in \mathfrak{p}^{j+1}$. Set $g = \varpi^l(z_2 - z_1)/(\varpi z_1 z_2) \in \mathfrak{p}^l$ and $e = f = 0$. Then there is a matrix identity

$$h(l,m)^{-1} \begin{bmatrix} I_2 & X \\ & I_2 \end{bmatrix} h(l,m) \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z_2\varpi && 1 \end{bmatrix} = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z_1\varpi && 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \frac{u_2}{u_1} && \frac{z_2-z_1}{\varpi z_1 z_2} \\ && 1 & \\ &&& \frac{u_1}{u_2} \end{bmatrix},$$

where the rightmost matrix lies in $K^{\#}(\mathfrak{p}^n)$. We therefore get the disjoint union

$$\bigcup_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n)$$

$$= \bigsqcup_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{\left[\frac{n-1}{2}\right]})/\mathfrak{p}^{n-1}} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n)$$

$$\sqcup \bigsqcup_{j=\max(n-m-1,0)}^{\left[\frac{n-3}{2}\right]} \bigsqcup_{u \in \mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} K_{l,m} \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & u\varpi^{j+1} && 1 \end{bmatrix} K^{\#}(\mathfrak{p}^n).$$

Summary

The following proposition summarizes our results in this section.

4.1 Proposition. *Let $l, m \geq 0$. The following are the disjoint double cosets in $K_{l,m}\backslash K^H/K^\#(\mathfrak{p}^n)$ that can possibly make a non-trivial contribution to the integral (10).*

$$\bigsqcup_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o}\cap\mathfrak{p}^{\left[\frac{n-1}{2}\right]})/\mathfrak{p}^{n-1}} K_{l,m}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}K^\#(\mathfrak{p}^n) \qquad \text{for } l,m\geq 0, \tag{41}$$

$$\bigsqcup_{j=\max(n-m-1,0)}^{\left[\frac{n-3}{2}\right]}\ \bigsqcup_{u\in\mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} K_{l,m}\begin{bmatrix}1&&&\\&1&&\\&&1&\\&u\varpi^{j+1}&&1\end{bmatrix}K^\#(\mathfrak{p}^n) \qquad \text{for } l,m\geq 0, \tag{42}$$

$$K_{l,m}s_1s_2s_1K^\#(\mathfrak{p}^n) \qquad \text{for } l\geq 0,\ m\geq n. \tag{43}$$

For $n = 1$ this reduces to

$$K_{l,m}K^\#(\mathfrak{p}) \qquad \text{for } l,m\geq 0, \tag{44}$$

$$K_{l,m}s_1s_2s_1K^\#(\mathfrak{p}) \qquad \text{for } l\geq 0,\ m\geq 1. \tag{45}$$

5 Volume computations

With a view towards the integral (10), we will now compute the volumes of the sets $K_{l,m}\backslash K_{l,m}AK^\#(\mathfrak{p}^n)$, where A is one of the representatives of the disjoint double cosets in (41), (42) or (43). As in Sect. 3.8 of [6], we have

$$\int\limits_{K_{l,m}\backslash K_{l,m}AK^\#(\mathfrak{p}^n)} dh = \mathrm{vol}(K^\#(\mathfrak{p}^n))\Big(\int\limits_{K_{l,m}\cap\left(AK^\#(\mathfrak{p}^n)A^{-1}\right)} dt\Big)^{-1}. \tag{46}$$

Note that

$$\mathrm{vol}(K^\#(\mathfrak{p}^n)) = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)} \tag{47}$$

from (19) and the fact that $\mathrm{vol}(K^H) = 1$. Hence we are reduced to computing

$$V(l,m,A) := \int\limits_{K_{l,m}\cap\left(AK^\#(\mathfrak{p}^n)A^{-1}\right)} dt. \tag{48}$$

5.1 Volume of double cosets (41) and (42)

In this case $A = \begin{bmatrix}1&&&\\&1&&\\&&1&\\&z\varpi&&1\end{bmatrix}$ for $z \in (\mathfrak{p}^{n-m-1}\cap\mathfrak{o})/\mathfrak{p}^{n-1}$. We want to find the volume of the set $h(l,m)^{-1}R(F)h(l,m)\cap AK^\#(\mathfrak{p}^n)A^{-1}$. Let $\nu(z) = j$ with $j \leq n-1$. Conjugation of $h(l,m)^{-1}R(F)h(l,m)\cap AK^\#(\mathfrak{p}^n)A^{-1}$ with an element of the form $\mathrm{diag}(1,1,u,u)$, where $u \in \mathfrak{o}^\times$, leaves $R(F)$ and $K^\#(\mathfrak{p}^n)$ unchanged, and results in replacing z by uz without any change in the volume. We may therefore assume that $z = \varpi^j$. Since $j \leq n-1$, it is clear that

$$AK^\#(\mathfrak{p}^n)A^{-1} \subset K^\#(\mathfrak{p}^{j+1}). \tag{49}$$

If we write an element of $R(F)$ as tn with $t = \begin{bmatrix} x+by/2 & yc \\ -ya & x-by/2 \end{bmatrix} \in T(F)$ and $n = \begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix}$, $X = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$, then (49) gives the following necessary condition for $h(l,m)^{-1}tnh(l,m) \in AK^{\#}(\mathfrak{p}^n)A^{-1}$,

$$\begin{bmatrix} x+by/2 & yc\varpi^{-m} \\ -ya\varpi^{m} & x-by/2 \end{bmatrix} \in \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{j+1} \\ \mathfrak{o} & \mathfrak{o}^\times \end{bmatrix} \subset \mathrm{GL}_2(\mathfrak{o}) \quad \text{and} \quad X \in \begin{bmatrix} \mathfrak{p}^{2m+l} & \mathfrak{p}^{m+l} \\ \mathfrak{p}^{m+l} & \mathfrak{p}^{l} \end{bmatrix}. \tag{50}$$

Set $B = A^{-1}h(l,m)^{-1}tnh(l,m)A$. We want to find further necessary conditions for $B \in K^{\#}(\mathfrak{p}^n)$. Looking at the $(4,2)$ coefficient of B, we get

$$\varpi^{-l}g(x+by/2)\varpi^{2+2j} \in \mathfrak{p}^n, \qquad \text{and hence} \qquad g \in \mathfrak{p}^{n-2-2j+l}. \tag{51}$$

Using the $(4,3)$ coefficient of B, we get

$$\varpi^{l}cy + \varpi^{j+1}f(x \pm by/2) \in \mathfrak{p}^{n+m+l}. \tag{52}$$

A direct computation shows that the conditions (50), (51) and (52) are also sufficient to conclude that $B \in K^{\#}(\mathfrak{p}^n)$. Note that $\varpi^{l}cy + \varpi^{j+1}f(x+by/2) \in \mathfrak{p}^{n+m+l}$ and $y \in \mathfrak{p}^{m+j+1}$ implies that $f \in \mathfrak{p}^{m+l}$ and $\varpi^{l}cy + \varpi^{j+1}f(x-by/2) \in \mathfrak{p}^{n+m+l}$. To summarize, the following are the necessary and sufficient conditions on t and n for $h(l,m)^{-1}tnh(l,m) \in AK^{\#}(\mathfrak{p}^n)A^{-1}$.

$$\begin{aligned} &y \in \mathfrak{p}^{m+j+1}, \qquad x \pm by/2 \in \mathfrak{o}^\times \\ &e \in \mathfrak{p}^{2m+l}, \qquad g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^{l}, \qquad \varpi^{l}cy + \varpi^{j+1}f(x+by/2) \in \mathfrak{p}^{n+m+l}. \end{aligned} \tag{53}$$

For fixed values of x, y satisfying the first two conditions, we are interested in

$$\begin{aligned} &\mathrm{vol}(\{(e,f,g) \in F^3 : e \in \mathfrak{p}^{2m+l},\ g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^{l},\ \varpi^{l}cy + \varpi^{j+1}f(x+by/2) \in \mathfrak{p}^{n+m+l}\}) \\ &\quad = \mathrm{vol}(\{(e,f,g) \in F^3 : e \in \mathfrak{p}^{2m+l},\ g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^{l},\ f \in \mathfrak{p}^{n+m+l-j-1} - \varpi^{l-j-1}cy(x+by/2)^{-1}\}) \\ &\quad = \mathrm{vol}(\{(e,f,g) \in F^3 : e \in \mathfrak{p}^{2m+l},\ g \in \mathfrak{p}^{n-2-2j+l} \cap \mathfrak{p}^{l},\ f \in \mathfrak{p}^{n+m+l-j-1}\}). \end{aligned}$$

Note that if $j \le \left[\frac{n-3}{2}\right]$, then $n-2-2j \ge 0$, and if $j \ge \left[\frac{n-1}{2}\right]$, then $n-2-2j \le 0$. Hence, the above volume is

$$\begin{aligned} &q^{-2n-3m-3l+3j+3} \quad \text{if} \quad j \le \left[\frac{n-3}{2}\right]; \\ &q^{-n-3m-3l+j+1} \quad \text{if} \quad j \ge \left[\frac{n-1}{2}\right]. \end{aligned} \tag{54}$$

By an argument similar to Lemma 3.8.3 of [6], we get

$$\mathrm{vol}(T(F) \cap \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{m+j+1} \\ \mathfrak{o} & \mathfrak{o}^\times \end{bmatrix})^{-1} = (1 - \left(\frac{L}{\mathfrak{p}}\right)q^{-1})q^{m+j+1}. \tag{55}$$

5.2 Volume of double coset (43)

In this case, we have $A = s_1s_2s_1$ and $m \ge n$. Note that

$$V(l,m,s_1s_2s_1) = \int\limits_{(h(l,m)^{-1}R(F)h(l,m)) \cap \left(s_1s_2s_1K^{\#}(\mathfrak{p}^n)(s_1s_2s_1)^{-1}\right)} dt.$$

We have

$$s_1s_2s_1K^{\#}(\mathfrak{p}^n)(s_1s_2s_1)^{-1} = K^H \cap \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{o}^\times & \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o}^\times & \mathfrak{p}^n \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o}^\times \end{bmatrix}. \tag{56}$$

We have to find the intersection of this compact group with $h(l,m)^{-1}R(F)h(l,m)$. Set

$$L_1 := \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o}^\times \end{bmatrix} \subset \mathrm{GL}_2(\mathfrak{o}), \quad N_1 := \{X \in \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n \end{bmatrix} : {}^tX = X\} \subset F^3.$$

Then L_1 and N_1 are the upper left and upper right blocks of (56), respectively. Write a given element of $R(F)$ as tn with $t \in T(F)$ and $n \in U(F)$. If $n = \begin{bmatrix} 1_2 & X \\ & 1_2 \end{bmatrix}$, then by arguments similar to those in Sect. 3.8 of [6], we see that tn lies in $s_1s_2s_1K^\#(\mathfrak{p}^n)(s_1s_2s_1)^{-1}$ if and only if

$$\begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} t \begin{bmatrix} \varpi^{m} & \\ & 1 \end{bmatrix} \in L_1 \tag{57}$$

and

$$\begin{bmatrix} \varpi^{-2m-l} & \\ & \varpi^{-m-l} \end{bmatrix} X \begin{bmatrix} 1 & \\ & \varpi^{m} \end{bmatrix} \in N_1. \tag{58}$$

It follows that

$$\begin{aligned} &\mathrm{vol}\big(\{X \in F^3 : \begin{bmatrix} \varpi^{-2m-l} & \\ & \varpi^{-m-l} \end{bmatrix} X \begin{bmatrix} 1 & \\ & \varpi^{m} \end{bmatrix} \in N_1\}\big) \\ &\quad = \mathrm{vol}\big(\{X \in F^3 : X \in \begin{bmatrix} \varpi^{2m+l} & \\ & \varpi^{m+l} \end{bmatrix} N_1 \begin{bmatrix} 1 & \\ & \varpi^{-m} \end{bmatrix}\}\big) \\ &\quad = q^{-3m-3l}\mathrm{vol}(N_1) = q^{-3m-3l-2n}. \end{aligned}$$

Let

$$T_m = \{t \in T(F) : \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} t \begin{bmatrix} \varpi^{m} & \\ & 1 \end{bmatrix} \in L_1\} = T(F) \cap \begin{bmatrix} \varpi^{m} & \\ & 1 \end{bmatrix} L_1 \begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}. \tag{59}$$

So far, we have $V(l,m,s_1s_2s_1)^{-1} = q^{3m+3l+2n}\mathrm{vol}(T_m)^{-1}$. By an argument similar to Lemma 3.8.4 of [6], and using $m \geq n$, we get

$$\mathrm{vol}(T_m)^{-1} = \Big(1 - \Big(\frac{L}{\mathfrak{p}}\Big)q^{-1}\Big)q^m. \tag{60}$$

The following proposition summarizes the volume computations in this section.

5.1 Proposition. *i) Let $m \geq 0$. Let $A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & z\varpi & & 1 \end{bmatrix}$ for $z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o})/\mathfrak{p}^{n-1}$ and set $\nu(z) = j$.*

If $j \leq \left[\frac{n-3}{2}\right]$, then

$$V_j^{l,m} := \int\limits_{K_{l,m}\backslash K_{l,m}AK^\#(\mathfrak{p}^n)} dh = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}(1 - \Big(\frac{L}{\mathfrak{p}}\Big)q^{-1})q^{2n+4m+3l-2j-2}, \tag{61}$$

and if $j \geq \left[\frac{n-1}{2}\right]$, then

$$V^{l,m} := \int\limits_{K_{l,m}\backslash K_{l,m}AK^\#(\mathfrak{p}^n)} dh = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}(1 - \Big(\frac{L}{\mathfrak{p}}\Big)q^{-1})q^{n+4m+3l}. \tag{62}$$

ii) For all $m \geq n$,

$$V_{s_1s_2s_1}^{l,m} := \int\limits_{K_{l,m}\backslash K_{l,m}s_1s_2s_1K^\#(\mathfrak{p}^n)} dh = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}\Big(1 - \Big(\frac{L}{\mathfrak{p}}\Big)q^{-1}\Big)q^{4m+3l+2n}. \tag{63}$$

iii) In particular, for $n = 1$,

$$\int_{K_{l,m}\backslash K_{l,m}K^{\#}(\mathfrak{p})} dh = \frac{q-1}{(q+1)(q^4-1)}(1 - \left(\frac{L}{\mathfrak{p}}\right)q^{-1})q^{4m+3l+1} \qquad (m \geq 0),$$

$$\int_{K_{l,m}\backslash K_{l,m}s_1s_2s_1K^{\#}(\mathfrak{p})} dh = \frac{q-1}{(q+1)(q^4-1)}\left(1 - \left(\frac{L}{\mathfrak{p}}\right)q^{-1}\right)q^{4m+3l+2} \qquad (m > 0).$$

Note that the right hand side of (62) is independent of j. This will play an important role in the evaluation of the zeta integral.

6 Main local theorem

In this section we will calculate the integral (10). From Proposition 4.1, we have

$$\begin{aligned} Z(s) = \sum_{l,m\geq 0} B(h(l,m))\Big(&\sum_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o}\cap\mathfrak{p}^{\left[\frac{n-1}{2}\right]})/\mathfrak{p}^{n-1}} W^{\#}(\eta h(l,m)A(z),s)V^{l,m} \\ &+ \sum_{j=\max(n-m-1,0)}^{\left[\frac{n-3}{2}\right]} \sum_{u\in\mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} W^{\#}(\eta h(l,m)A(\varpi^{j+1}u),s)V_j^{l,m}\Big) \\ + \sum_{l\geq 0,m\geq n} B(h(l,m))W^{\#}(\eta h(l,m)s_1s_2s_1,s)V^{l,m}_{s_1s_2s_1} \end{aligned} \tag{64}$$

where $A(z) = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ & z\varpi && 1 \end{bmatrix}$. By (9), (29) and (31) we get

$$W^{\#}(\eta h(l,m)A(z),s) = |\varpi^{2m+l}|^{3(s+\frac{1}{2})}\omega_\pi(\varpi^{-2m-l})\omega_\tau(\varpi^{-m-l})W^{(0)}(\begin{bmatrix} \varpi^l & 0 \\ \varpi z & 1 \end{bmatrix}), \tag{65}$$

$$W^{\#}(\eta h(l,m)s_1s_2s_1,s) = |\varpi^{2m+l}|^{3(s+\frac{1}{2})}\omega_\pi(\varpi^{-2m-l})\omega_\tau(\varpi^{-m-l})W^{(0)}(\begin{bmatrix} & \varpi^l \\ -1 & \end{bmatrix}). \tag{66}$$

Set $C_{l,m} := |\varpi^{2m+l}|^{3(s+\frac{1}{2})}\omega_\pi(\varpi^{-2m-l})\omega_\tau(\varpi^{-m-l})$. Substituting (65) and (66) into (64), we get

$$\begin{aligned} Z(s) = &\sum_{l\geq 0} B(h(l,0))C_{l,0}W^{(0)}(\begin{bmatrix} \varpi^l & 0 \\ 0 & 1 \end{bmatrix})V^{l,0} \\ &+ \sum_{l\geq 0,m\geq 1} B(h(l,m))C_{l,m}V^{l,m}\Big(\sum_{z\in(\mathfrak{p}^{n-m-1}\cap\mathfrak{o}\cap\mathfrak{p}^{\left[\frac{n-1}{2}\right]})/\mathfrak{p}^{n-1}} W^{(0)}(\begin{bmatrix} \varpi^l & 0 \\ \varpi z & 1 \end{bmatrix})\Big) \\ &+ \sum_{l\geq 0,m\geq 1} B(h(l,m))C_{l,m}\Big(\sum_{j=\max(n-m-1,0)}^{\left[\frac{n-3}{2}\right]} \sum_{u\in\mathfrak{o}^\times/(1+\mathfrak{p}^{j+1})} W^{(0)}(\begin{bmatrix} \varpi^l & 0 \\ \varpi^{j+1}u & 1 \end{bmatrix})V_j^{l,m}\Big) \\ &+ \sum_{l\geq 0,m\geq n} B(h(l,m))C_{l,m}W^{(0)}(\begin{bmatrix} & \varpi^l \\ -1 & \end{bmatrix})V^{l,m}_{s_1s_2s_1}. \end{aligned} \tag{67}$$

If $n = 1$, then the inner sum over z in the second term above reduces to just $z = 0$, and the third term above is not present. In this case, the integral $Z(s)$ was computed in Theorem 3.9.1 of [6].

From now on we will assume that $n \geq 2$. As the following lemma shows, the fact that the representation τ has conductor $\mathfrak{p}^n$ implies that the middle two expressions in formula (67) are zero.

6.1 Lemma. *Let $m \geq 1$ and $n \geq 2$.*

i) For any $g \in \mathrm{GL}_2(F)$,

$$\sum_{z \in (\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{\left[\frac{n-1}{2}\right]})/\mathfrak{p}^{n-1}} W^{(0)}(g \begin{bmatrix} 1 & 0 \\ \varpi z & 1 \end{bmatrix}) = 0.$$

ii) For $2j+2 < n$ and any z with $\nu(z) = j$,

$$W^{(0)}(\begin{bmatrix} \varpi^l & 0 \\ \varpi z & 1 \end{bmatrix}) = 0.$$

Proof: i) Let $t = \max(n-m-1, 0, \left[\frac{n-1}{2}\right])$. We have $\mathfrak{p}^{n-m-1} \cap \mathfrak{o} \cap \mathfrak{p}^{\left[\frac{n-1}{2}\right]} = \mathfrak{p}^t$ and, since $m \geq 1$ and $n \geq 2$, we see that $t+1 < n$. Define $\hat{W}(g) = \sum\limits_{z \in \mathfrak{p}^{t+1}/\mathfrak{p}^n} W^{(0)}(g \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}) \in V_\tau$. Then, by definition $\hat{W}$ is right invariant under the group $\begin{bmatrix} 1 & 0 \\ \mathfrak{p}^{t+1} & 1 \end{bmatrix}$. Since $W^{(0)}$ is right invariant under $K^{(1)}(\mathfrak{p}^n)$, we see that $\hat{W}$ is right invariant under $\begin{bmatrix} \mathfrak{o}^\times & \\ & \mathfrak{o}^\times \end{bmatrix}$. The matrix identity $\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ (1+xz)^{-1}z & 1 \end{bmatrix} \begin{bmatrix} 1+xz & x \\ & (1+xz)^{-1} \end{bmatrix}$ and $1 + xz \in \mathfrak{o}^\times$ implies that $\hat{W}$ is also right invariant under the group $\begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix}$. Since $\begin{bmatrix} 1 & 0 \\ \mathfrak{p}^{t+1} & 1 \end{bmatrix}$, $\begin{bmatrix} \mathfrak{o}^\times & \\ & \mathfrak{o}^\times \end{bmatrix}$ and $\begin{bmatrix} 1 & \mathfrak{o} \\ & 1 \end{bmatrix}$ generate $K^{(1)}(\mathfrak{p}^{t+1})$, it follows that $\hat{W}$ is a vector in V_τ that is right invariant under $K^{(1)}(\mathfrak{p}^{t+1})$. Since τ has level $\mathfrak{p}^n$ and $t+1 < n$, this implies $\hat{W} = 0$, as claimed.

ii) Let z_1, z_2 be such that $\nu(z_1) = \nu(z_2) = j$ and $z_1/z_2 \in 1 + \mathfrak{p}^{j+1}$. Consider the matrix identity

$$\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\varpi^l(z_2-z_1)}{\varpi z_1 z_2} \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z_2 & 1 \end{bmatrix} \begin{bmatrix} \frac{z_1}{z_2} & \frac{(z_2-z_1)}{\varpi z_1 z_2} \\ & \frac{z_2}{z_1} \end{bmatrix}.$$

Since the additive character ψ is trivial on $\mathfrak{o}$ and the rightmost matrix is in $K^{(1)}(\mathfrak{p}^n)$, it implies that

$$W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix}) = W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z u & 1 \end{bmatrix}) \tag{68}$$

for every $u \in 1 + \mathfrak{p}^{j+1}$ and $z \in \mathfrak{o}$ with $\nu(z) = j$ (we have essentially derived the well-definedness of the third sum in (67)). Writing $u = 1 + b\varpi^{j+1}$ with $b \in \mathfrak{o}$ and integrating both sides of (68) with respect to b, we get

$$\begin{aligned} W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix}) &= \int\limits_{\mathfrak{o}} W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z b \varpi^{j+1} & 1 \end{bmatrix}) db \\ &= \int\limits_{\mathfrak{o}} W^{(0)}(\begin{bmatrix} \varpi^l & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi z & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ b\varpi^{2j+2} & 1 \end{bmatrix}) db. \end{aligned}$$

This last expression is zero, since $2j+2 < n$ and $\tilde{W}(g) := \int\limits_{\mathfrak{o}} W^{(0)}(g \begin{bmatrix} 1 & \\ b\varpi^{2j+2} & 1 \end{bmatrix}) db \in V_\tau$ is right invariant under $K^{(1)}(\mathfrak{p}^{2j+2})$. This concludes the proof. ∎

Using this lemma, (67) now becomes

$$Z(s) = \sum_{l \geq 0} B(h(l,0)) |\varpi^l|^{3(s+\frac{1}{2})} \omega_\pi(\varpi^{-l}) \omega_\tau(\varpi^{-l}) W^{(0)}(\begin{bmatrix} \varpi^l & 0 \\ 0 & 1 \end{bmatrix}) V^{l,0}$$

$$+\sum_{l\geq 0,m\geq n} B(h(l,m))|\varpi^{2m+l}|^{3(s+\frac{1}{2})}\omega_\pi(\varpi^{-2m-l})\omega_\tau(\varpi^{-m-l})W^{(0)}(\begin{bmatrix} & \varpi^l \\ -1 & \end{bmatrix})V^{l,m}_{s_1s_2s_1}. \tag{69}$$

Since $\begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix}$ normalizes $K^{(1)}(\mathfrak{p}^n)$, the vector $W'(g) := W^{(0)}(g\begin{bmatrix} & 1 \\ \varpi^n & \end{bmatrix})$ is another element of V_τ that is right invariant under $K^{(1)}(\mathfrak{p}^n)$. Since the space of vectors in V_τ right invariant under $K^{(1)}(\mathfrak{p}^n)$ is one dimensional, there is a constant $c \in \mathbb{C}$ such that $W^{(0)} = cW'$ (one can check that $c^{-2} = \omega_\tau(\varpi^n)$). Hence,

$$W^{(0)}(\begin{bmatrix} & \varpi^l \\ -1 & \end{bmatrix}) = cW^{(0)}(\begin{bmatrix} \varpi^{l+n} & \\ & -1 \end{bmatrix}) = cW^{(0)}(\begin{bmatrix} \varpi^{l+n} & \\ & 1 \end{bmatrix}). \tag{70}$$

This shows that in order to evaluate (69) we need the formula for the new-vector of τ in the Kirillov model. The possibilities for our generic, irreducible, admissible representation τ of $\mathrm{GL}_2(F)$ with unramified central character and conductor $\mathfrak{p}^n$, $n \geq 2$, are as follows. Either τ is a principal series representation $\chi_1 \times \chi_2$, where χ_1 and χ_2 are ramified characters of $F^\times$ (with $\chi_1\chi_2$ unramified); or $\tau = \chi\mathrm{St}_{\mathrm{GL}(2)}$, a twist of the Steinberg representation by a ramified character χ (such that χ^2 is unramified); or τ is supercuspidal. In each case the newform in the Kirillov model is given by the characteristic function of $\mathfrak{o}^\times$; see, e.g., [8]. It follows that all the terms in (70) are zero. The integral (69) reduces to

$$Z(s) = V^{0,0} = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}(1-\Big(\frac{L}{\mathfrak{p}}\Big)q^{-1})q^n. \tag{71}$$

Thus, we have proved the following result.

6.1 Theorem. *Let π be an unramified, irreducible, admissible representation of $\mathrm{GSp}_4(F)$ (not necessarily with trivial central character), and let τ be an irreducible, admissible, generic representation of $\mathrm{GL}_2(F)$ with unramified central character and conductor $\mathfrak{p}^n$ with $n \geq 2$. Let $Z(s)$ be the integral (3), where $W^\#$ is the function defined in Sect. 2, and B is the spherical Bessel function defined in Sect. 1 (ix). Then*

$$Z(s) = \frac{q-1}{q^{3(n-1)}(q+1)(q^4-1)}(1-\Big(\frac{L}{\mathfrak{p}}\Big)q^{-1})q^n. \tag{72}$$

Remark: For any unramified, irreducible, admissible representation π of $\mathrm{GSp}_4(F)$ and any of the representations τ of $\mathrm{GL}_2(F)$ mentioned in the theorem we have $L(s, \pi\times\tau) = 1$. Hence, up to a constant, the integral $Z(s)$ represents the L-factor $L(s, \pi \times \tau)$.

7 Global integral and special value of L-function

Let $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z})$. For a positive integer l denote by $S_l(\Gamma_2)$ the space of Siegel cusp forms of degree 2 and weight l with respect to Γ_2. Let $\Phi \in S_l(\Gamma_2)$ be a Hecke eigenform. It has a Fourier expansion

$$\Phi(Z) = \sum_{S>0} a(S,\Phi)e^{2\pi i \mathrm{tr}(SZ)},$$

where S runs through all symmetric, semi-integral, positive definite matrices of size two. Let us make the following two assumptions about the function Φ.

Assumption 1: $a(S,\Phi) \neq 0$ for some $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ such that $b^2 - 4ac = -D < 0$ where $-D$ is the discriminant of the imaginary quadratic field $L := \mathbb{Q}(\sqrt{-D})$.

Assumption 2: The weight l is a multiple of $w(-D)$, the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$. We have

$$w(-D) = \begin{cases} 4 & \text{if } D = 4, \\ 6 & \text{if } D = 3, \\ 2 & \text{otherwise.} \end{cases}$$

We lift the function Φ to a function ϕ_Φ on $H(\mathbb{A}) = \mathrm{GSp}_4(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$, in a standard way; see (141) in [6]. Let V_Φ be the automorphic representation generated by ϕ_Φ, and let $\pi_\Phi \cong \otimes'_p \pi_p$ be an irreducible component. Let $\Lambda = \otimes \Lambda_p$ be a character of $L^\times$ depending on S as constructed in Sect. 5.1 of [6].

Let $N = \prod p^{n_p} \in \mathbb{N}$. Denote the space of Maaß cusp forms of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$ by $S^M_{l_1}(N)$. A function $f \in S^M_{l_1}(N)$ has the Fourier expansion

$$f(x+iy) = \sum_{n \neq 0} a_n W_{\mathrm{sgn}(n)\frac{l_1}{2}, \frac{ir}{2}}(4\pi|n|y)e^{2\pi inx}, \tag{73}$$

where $W_{\nu,\mu}$ is a classical Whittaker function and $(\Delta_{l_1} + \lambda)f = 0$ with $\lambda = 1/4 + (r/2)^2$. Here Δ_{l_1} is the Laplace operator defined in Sect. 5.3 of [6]. Let $f \in S^M_{l_1}(N)$ be a Hecke eigenform.

If $ir/2 = (l_2 - 1)/2$ for some integer $l_2 > 0$, then the cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ constructed below is holomorphic at infinity of lowest weight l_2. In this case we make the additional assumptions that $l_2 \le l$ and $l_2 \le l_1$, where l is the weight of the Siegel cusp form Φ. Starting from f, we obtain another Maaß form $f_l \in S^M_l(N)$ by applying the raising and lowering operators as in (147) of [6]. Define a function $\hat{f}$ on $\mathrm{GL}_2(\mathbb{A})$ by

$$\hat{f}(\gamma_0 m k_0) = \Big(\frac{\gamma i + \delta}{|\gamma i + \delta|}\Big)^{-l} f_l\Big(\frac{\alpha i + \beta}{\gamma i + \delta}\Big), \tag{74}$$

where $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$, $m = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, $k_0 \in \prod_{p|N} K^{(1)}(\mathfrak{p}^{n_p}) \prod_{p \nmid N} \mathrm{GL}_2(\mathbb{Z}_p)$. Here, for $p|N$ we have $K^{(1)}(\mathfrak{p}^{n_p}) = \mathrm{GL}_2(\mathbb{Q}_p) \cap \begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ \mathfrak{p}^{n_p} & \mathbb{Z}_p^\times \end{bmatrix}$ with $\mathfrak{p} = p\mathbb{Z}_p$. Let $\tau_f \cong \otimes'_p \tau_p$ be the irreducible, cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $\hat{f}$. As in Sect. 5.2 of [6], define an Eisenstein series on $\mathrm{GU}(2,2;L)(\mathbb{A})$ by

$$E_\Lambda(g,s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_\Lambda(\gamma g, s) \tag{75}$$

where f_Λ is as defined in (154) of [6] from $\hat{f}$. We consider the global integral

$$Z(s,\Lambda) = \int\limits_{Z_H(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} E_\Lambda(h,s)\bar{\phi}(h)dh. \tag{76}$$

Now, applying Theorem 3.7 from [5], Theorem 4.4.1 from [6] and Theorem 6.1, we get

7.1 Theorem. *Let $\Phi \in S_l(\Gamma_2)$ be a cuspidal Siegel eigenform of degree 2 and even weight l satisfying the two assumptions above. Let $L = \mathbb{Q}(\sqrt{-D})$, where D is as in Assumption 1. Let $N = \prod p^{n_p}$ be a positive integer. Let f be a Maaß Hecke eigenform of weight $l_1 \in \mathbb{Z}$ with respect to $\Gamma_0(N)$. If f lies in a holomorphic discrete series with lowest weight l_2, then assume that $l_2 \le l$. Then the integral (76) is given by*

$$Z(s,\Lambda) = \kappa_\infty \kappa_N \frac{L(3s+\frac{1}{2}, \pi_\Phi \times \tau_f)}{\zeta(6s+1)L(3s+1, \tau_f \times \mathcal{AI}(\Lambda))}, \tag{77}$$

where

$$\kappa_\infty = \frac{1}{2}\overline{a(\Lambda)}c(1)\pi D^{-3s-\frac{l}{2}}\,(4\pi)^{-3s+\frac{3}{2}-l}\,\frac{\Gamma(3s+l-1+\frac{ir}{2})\Gamma(3s+l-1-\frac{ir}{2})}{\Gamma(3s+\frac{l+1}{2})},$$

$$\kappa_N = \prod_{p|N} \frac{p-1}{p^{3(n_p-1)}(p+1)(p^4-1)}(1-\Big(\frac{L}{p}\Big)p^{-1})p^{n_p}(1-p^{-6s-1})^{-1} \prod_{p^2|N} L_p(3s+1, \tau_p \times \mathcal{AI}(\Lambda_p)).$$

Here, the non-zero constant $c(1)$ is given by Eqn. (148) of [6], the non-zero constant $a(\Lambda)$ is defined in Sect. 5.1 of [6], and

$$\Big(\frac{L}{p}\Big) = \begin{cases} -1 & \text{if } p \text{ is inert in } L, \\ 0 & \text{if } p \text{ ramifies in } L, \\ 1 & \text{if } p \text{ splits in } L. \end{cases}$$

The quantity $\frac{ir}{2}$ is as in (73).

One would like to know if the local L-function $L_p(s, \tau_p \times \mathcal{AI}(\Lambda_p))$ is 1. If p is an odd prime, then the only case where the L-function $L_p(s, \tau_p \times \mathcal{AI}(\Lambda_p))$ is not identically 1 is when $p|D$, $\nu_p(N) = 2$ and τ_p is a certain induced representation or a certain twist of the Steinberg representation. The main difficulty is that if $\nu_p(N) = 2$, then it is not clear if the corresponding representation τ_p is induced or special or supercuspidal.

Let $\Gamma^{(2)}(N) := \{g \in \mathrm{Sp}_4(\mathbb{Z}) : g \equiv 1 \pmod N\}$ be the principal congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$. Let us denote the space of all Siegel cusp forms of weight l with respect to $\Gamma^{(2)}(N)$ by $S_l(\Gamma^{(2)}(N))$. For a Hecke eigenform $\Phi \in S_l(\Gamma^{(2)}(N))$, let $\mathbb{Q}(\Phi)$ be the subfield of $\mathbb{C}$ generated by all the Hecke eigenvalues of Φ. Let $S_l(\Gamma^{(2)}(N), \mathbb{Q}(\Phi))$ be the subspace of $S_l(\Gamma^{(2)}(N))$ consisting of cusp forms whose Fourier coefficients lie in $\mathbb{Q}(\Phi)$. For more details on this space we refer to Sect. 5.4 of [6]. Note that all the arguments of Sect. 5.4 of [6] are valid, with minor modifications, if we remove the restriction that N is square-free and use the definition (7) for $K^{\#}(\mathfrak{P}^n)$. Hence, we get the following special value result.

7.2 Theorem. *Let Φ be a cuspidal Siegel eigenform of weight l with respect to Γ_2 satisfying the two assumptions above, let D be as in Assumption 1 and let $\Phi \in S_l(\Gamma^{(2)}(N), \mathbb{Q}(\Phi))$. Let $N = \prod p^{n_p}$ be odd such that if $p|D$ then $n_p \neq 2$. Let Ψ be a normalized, holomorphic, cuspidal eigenform of weight l with respect to $\Gamma_0(N)$. Then*

$$\frac{L(\frac{l}{2} - 1, \pi_\Phi \times \tau_\Psi)}{\pi^{5l-8} \langle \Phi, \Phi \rangle \langle \Psi, \Psi \rangle_1} \in \bar{\mathbb{Q}}, \tag{78}$$

where $\langle \Phi, \Phi \rangle$ and $\langle \Psi, \Psi \rangle_1$ denote the Petersson inner products of Φ and Ψ, respectively, and $\bar{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Note that we have the above restriction on N because if $p|D$ and $n_p = 2$, then we do not know if the term $L_p((l-1)/2, \tau_p \times \mathcal{AI}(\Lambda_p))$ in κ_N from Theorem 7.1 is algebraic or not.

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